

Analysis of Jump Behavior in Nonlinear Electronic Circuits Using Computational Geometric Methods

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Abstract—In this work we describe the behavior of electrical circuits by a mixture of algebraic and differential equations. We show how to use a geometric interpretation and geometric algorithms to explicitly compute operation points for a special class of electronic circuits. To that end, we discuss how to trace curves on folded manifolds.

I. INTRODUCTION

The concept of electrical circuits was first presented by Kirchhoff and Maxwell; for a historical study of network analysis and further references see Mathis [3]. Such circuits are characterized by a certain class of so-called network elements and a connection port. The essential elements are linear and non-linear resistors, inductors and capacitors as well as dependent and independent sources which are described by very simple constitutive relations. To connect these network elements we use a connection b -port which conserves the energy; this idea was first presented by Belevitch. In elementary considerations such a b -port is described by Kirchhoff's laws and the constitutive relations of ideal transformers. A complete discussion of the general class of nonlinear RCL circuits based on the ideas of Belevitch is given by Mathis [3] (see also Mathis and Marten [5]). Although it is possible to set up explicit ordinary differential equations (ODE) or so-called state-space equations $\dot{x} = f(x, t)$ $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ the resulting dynamical equations for describing networks are not well adapted to the process of derivation because the constitutive relations of the network elements and the description equations of the connection b -port consist of a mixture of algebraic and differential equations. Therefore a more general concept of circuit description is needed.

The paper is organized in the following manner. In the next section we present the main ideas of circuit analysis using concepts from differential geometry. Then we show that this approach is not only suitable for abstract circuit analysis but it is also an appropriate basis for studying jump phenomena of a certain class of electronic circuits using concepts from computational differential geometry. Our ideas are illustrated by a simple example. However, it is applicable to more interesting classes of circuits including so-called resonance tunneling devices (RTD) as well as bipolar and MOS transistors.

II. ELECTRICAL CIRCUITS AND DIFFERENTIAL GEOMETRY

It was observed for the first time by Moser that the property of reciprocity is crucial for the formulation of a theory of nonlinear electrical circuits including dissipation of energy. His idea was generalized by Brayton and Moser [1] in 1964 where the differential equations for the class of nonlinear reciprocal circuits were formulated in a systematic manner. From a differential-geometric point of view, further mathematical structures had to be defined based on the constitutive relations for the network elements. This was done by Smale [2] in 1972 for the first time and generalized by Matsumoto to a more general class of nonlinear circuits in 1975.

The dynamics of a system and an electrical circuit can be formulated by a set of nonlinear differential equations with respect to currents and voltages where certain algebraic constraints have to be added. From the classical point of view this means that the collection of differential equations for the capacitors and inductors (constitutive relations) have to be combined with Kirchhoff's laws (homogeneous linear algebraic equations) and the resistive constitutive relations (nonlinear equations). In the framework of differential geometry we have to consider (nonlinear) differential equations on the state space that is endowed in generic cases with the structure of a differentiable manifold. Therefore we have to construct the state space \mathcal{S} and the vector field X in order to define the dynamics of a circuit: $\dot{\xi} = X \circ \xi$ $X: \mathcal{S} \rightarrow T\mathcal{S}$. The state space \mathcal{S} can be constructed using the Kirchhoff space $\mathcal{K} \subset \mathbb{R}^b \times \mathbb{R}^{b^*}$ and the set of zeros of F_R , the resistive equations. If the condition of transversality of these subsets is fulfilled their intersection is a differentiable manifold (Smale [2]). For the construction of the vector field X a 2-tensor $g(\cdot, \cdot): \mathcal{S} \rightarrow T^* \mathcal{S} \otimes T^* \mathcal{S}$ and a 1-form $\omega: \mathcal{S} \rightarrow T^* \mathcal{S}$ can be obtained from the constitutive relations and Kirchhoff's laws. It was shown by Brayton and Moser [1] that ω can be obtained from a so-called mixed potential P using the exterior derivative, that is $\omega = dP$. We get the 2-tensor g if a 2-tensor G is defined on the linear subspace of inductor currents and capacitor voltages

$$G := \sum_k L(i_L^k) di_L^k \otimes di_L^k - \sum_k C(u_C^k) du_C^k \otimes du_C^k \quad (1)$$

and pullback this 2-tensor on the state space, that is $g = \pi^*G$ where π is a projection from \mathcal{S} to the linear subspace of inductor currents and capacitor voltages. In the same manner the 1-form ω can be obtained. Using these objects an abstract equation for the vector field X can be formulated $g(X, Y) = \omega(Y)$ for all $Y \in T\mathcal{S}$. If this equation has a unique (local) solution the case of a (local) generic circuit dynamics is characterized. The condition for the (local) existence of X is that g is non-degenerated, that is if G is non-degenerated and π^* exists. These conditions can be translated in a more concrete manner and it can be shown that with a suitable “disturbance” of the constitutive relations, the two conditions are fulfilled – this is called generic. For more details and numerical aspects of circuit analysis see Mathis [3] and [4].

III. COMPUTATIONAL GEOMETRIC METHODS

A. Initial Situation

Opposed to conventional methods which are using homotopy methods to search operation points, here homotopy method are only used in finding specific starting points on the manifold. After obtaining a starting point, we use the dynamic on the manifold to trace a solution. In this approach we will consider the derived circuit equations as a geometrical problem only.

As mentioned, we can treat state space as a differentiable manifold and the dynamic defined on it as a differential equation system:

$$B(x)\dot{x} = f(x, y, t), \quad (2)$$

$$0 = g(x, y), \quad (3)$$

where x and y are the vectors of the electrical quantities, t a time variable and B a matrix. The state space manifold is generally embedded in a space of higher dimension. We will restrict our study to manifolds of dimension one or two and co-dimension one, e. g. curves in the plane or surfaces in Euclidean 3-space. The dynamic will then generate curves on that manifold, which we want like to trace. Notice, that in general it is not possible to obtain a closed explicit form for the manifold.

B. Example: Van-Der-Pol-Oscillator

The degenerated Van-der-Pol-Oscillator is a simple circuit consisting of a resistor and a capacitor that are connected in a circle, described by:

$$\frac{dv}{dt} = i \quad (4)$$

$$0 = -v - i^3 + i \quad (5)$$

where the differential equation (4) characterizes the capacity and the non-linear relation (5) defines the resistance. From a geometric point of view, the solution of (5) is a one-dimensional manifold, e. g., a curve M in the plane and the differential equation generates a dynamic which should be solved with respect to the current i . But this is not feasible globally, since in the extrema of the curve wrt. v , the dynamic

degenerates to 0. Since $i \neq 0$ in these points, they can not be equilibria and therefore the model does not capture the behavior of the circuit.

The described problem can be solved by a so-called Tichonov regularisation [8], [9] which transforms the algebraic equation to a differential equation

$$\varepsilon \frac{di}{dt} = -v - i^3 + i. \quad (6)$$

with ε near zero. The dynamic of the system is now generically smooth and the formerly singular points now exhibit a very fast dynamic, the system “jumps” from a formerly singular point tangentially to another area of the manifold. We want to capture this phenomenon with differential geometric tools and trace the curve on the manifold to an extremum where it jumps tangentially, thus following the oscillating path.

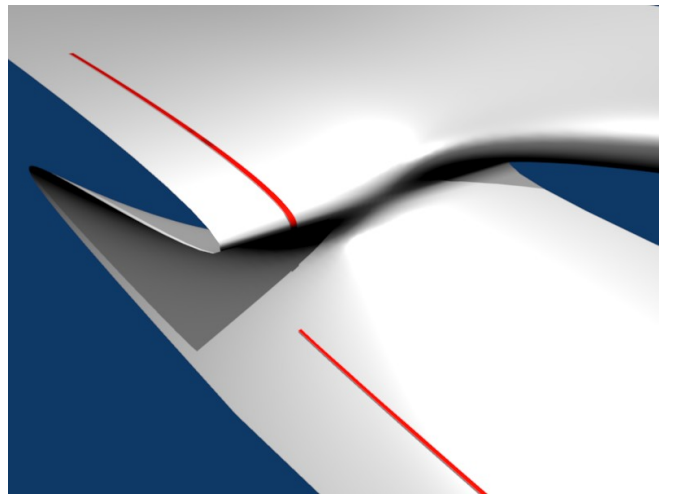


Fig. 1. Possible curves jump on a folded manifold.

C. Basic Principles of Tracing Curves on Two-Dimensional Manifolds Embedded in Three-Dimensional Space

To start tracing a curve on a manifold, we first need a starting point on this manifold. For that we can, e. g., use standard homotopy methods which are discussed in the next section.

We can then trace a curve on the manifold by numerically integrating the given differential equations which describe a tangent vector field on the manifold. This should lead us to an operation point or, if an oscillating circuit is given, represents the set of states.

Following the curve, we want to consider what happens, when the curve reaches a fold, i. e., a generalized extremum situation on the manifold. This will be the case if the circuit oscillates: the operation point jumps from the extremum to another (non-neighboring) point of the manifold. Therefore we’re interested in a submanifold S_A of maximum points, as they characterize points where a jump may start. Additionally, we want to determine from the submanifold (S_A) a second one

by orthogonal projection(S_B) that represents the set of points where a jump can ends.

To trace a path on a two-dimensional manifold M embedded in a three-dimensional space we can define a differentiable manifold by:

$$g(x, y, z) = 0 \quad (7)$$

to define a path on that manifold we use the parameter t :

$$g(x(t), y(t), z(t)) \quad (8)$$

Supposing we need the set of maximum points in z -direction, we differentiate wrt. z by:

$$g_z(x, y, z) = 0 \quad (9)$$

Differentiate wrt. t leads to:

$$\frac{d}{dt}g_z(x(t), y(t), z(t)) = (g_{zx}, g_{zy}, g_{zz}) \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = 0 \quad (10)$$

Using the nabla-operator as a shorthand we get

$$\vec{\nabla}g_z \begin{pmatrix} x'(t) \\ y'(t) \\ z'(t) \end{pmatrix} = 0 \quad (11)$$

In this simple setup, we can easily find a starting point by using z as a function of x and y :

$$\begin{aligned} g(x(t), y(t), \lambda(x(t), y(t))) &= 0 \\ \Rightarrow \frac{d}{dt}g(x(t), y(t), \lambda(x(t), y(t))) &= 0 \\ \Rightarrow (g_x, g_y, g_z) \begin{pmatrix} x' \\ y' \\ \lambda' \end{pmatrix} &= 0 \end{aligned} \quad (12)$$

With it we are able to calculate λ .

As an example we see in Figures 1 and 2 the implicit function:

$$x = -z^3 + yz^2$$

The shape can be taken as an example set of operation points.

D. Using Homotopy Methods for Finding Starting Points

The methods described in the previous section assume that a starting point can easily be found. This is not true for manifolds with a co-dimension greater than one. To acquire such a point it is possible to use homotopy methods established by Kellogg et al. [10], Smale [11], and Chow et al. [12] which have become a powerful tool in finding solutions of various nonlinear problems, such as zeros or fixed points of maps. A distinctive advantage of the homotopy method is that the algorithm generated by it exhibits the global convergence under weaker conditions. A good introduction is given by Allgower and Georg [13]. The homotopy concept is used to determine solutions of high-dimensional non-linear equation systems by initially finding a solution to a simpler problem and then systematically transforming it to the actual problem by embedding it in a homotopy.

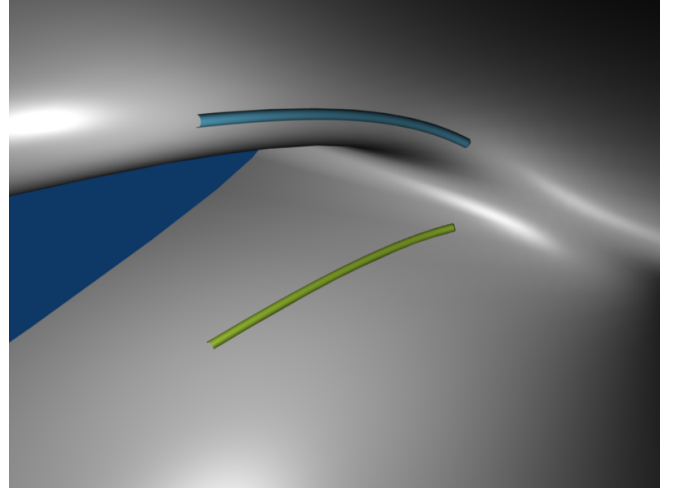


Fig. 2. Traced curve of maximal turning points (blue) and its projection on the same manifold (green)

Consider $g(w)$ is regular. The implicit function theorem implies the existence of a curve p which solves:

$$H(w, \lambda, w_0) = g(w) + (\lambda - 1)g(w_0), \quad \lambda \in [0, 1] \quad (13)$$

If we formulate the map $g(w)$ so that its zero set is a point on the manifold, the curve p will converge towards it under certain conditions. This method was used by Naß and Wolter [6] [7] to find solutions for similar geometric problems. By using this experience this method can be applied to higher dimensions.

IV. CONCLUSION

We have shown, how geometrical algorithms can be used to solve the problem of finding operation points of a class of oscillating electronic circuits. Basically we show how to explicitly calculate “jump” sets on the state space manifold that capture the behavior of the circuit. If it is possible to extend these methods to higher dimensional spaces, they can potentially be used as an alternative to SPICE-based circuit simulators in commercial software packages. To extend the methods to higher dimensions it is necessary and important to create new forms of representation, e. g., animation, color, texture, or dimensional reduction to present and debug the results.

REFERENCES

- [1] R.K. Brayton and J.K. Moser, *A Theory of Nonlinear Networks I+II*. *Quartly Appl. Math.* 22 (1964) pp. 1-33, 81-104.
- [2] S. Smale, *On the mathematical foundation of electrical circuit theory*. *Journ. Differential Geometry* 7(1972) pp. 193-210.
- [3] W. Mathis, *Theorie nichtlinearer Netzwerke*. Springer Verlag, Berlin-New York 1987.
- [4] W. Mathis, *Recent Developments in Numerical Integration of Differential Equations*. *Intern. Journ. Num. Mod.* 7(1994) pp. 99-125.
- [5] W. Marten, L.O. Chua and W. Mathis, *On the geometrical meaning of pseudo hybrid and mixed potential*. *Intern. Journ. Electron. Commun. (AEÜ)*, 46 (1992) pp. 305-309.

- [6] Henning Na and Franz-Erich Wolter and Hannes Thielhelm and Cem Dogan, *Medial Axis Inverse Transform in 3-Dimensional Riemannian Complete Manifolds*. Proceedings of NASAGEM, Hannover, 26th October 2007 10(2007)
- [7] Henning Na and Franz-Erich Wolter and Cem Dogan and Hannes Thielhelm, *Computation of Geodesic Voronoi Diagrams in 3-Space using Medial Equations*. Proceedings of NASAGEM, 26th October 2007 10(2007)
- [8] E.F. Mischenko and N.Kh. Rozov, *Differential Equations with Small Parameters and Relaxation Oscillations*. Plenum Press (1980).
- [9] Wolfgang Mathis, *Theorie nichtlinearer Netzwerke*. Springer (1987).
- [10] R.B. Kellogg, T.Y. Li and J.A. Yorke, *A constructive proof of the Brouwer fixed-point theorem and computational results*, SIAM J. Numer. Anal. 18 (1976), pp. 473483.
- [11] S. Smale, *A convergent process of price adjustment and global Newton method*, J. Math. Econ. 3 (1976), pp. 107120.
- [12] S.N. Chow, J. Mallet-Paret and J.A. Yorke, *Finding zeros of maps: Homotopy methods that are constructive with probability one*, Math. Comput. 32 (1978), pp. 887899.
- [13] E.L. Allgower and K. Georg, *Numerical Continuation Methods: An Introduction*, Springer-Verlag, Berlin (1990).