A Numerical Approach for Nonlinear Dynamical Circuits with Jumps

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***Abstract*—A special class of nonlinear electronic circuits exhibits jump effects which occurs e.g. when the state space of an electronic system contains a fold. This can lead to difficulties during the simulation of these systems with standard circuit simulators. A common treatment to overcome these problems is to regularise these systems by adding suitably located parasitic inductors L's and capacitors C's. We propose a geometric approach and derive methods for solving circuit equations by means of algorithms from computational differential geometry. In this paper the geometric methods will be applied to an electronic multivibrator and numerical results will be presented.**

I. Introduction

In this work our focus lies on electronic circuits with fast switching behaviour, i.e. circuits with seemingly discontinuous changes in state which we call "jumps" in state space. These jumps occur when the state space of an electronic system contains a fold. Properties which indicate fast switching behaviour are e.g. topological properties like positive feedback and circuit characteristics like negative differential resistance or port characteristics exhibiting a negative differential slope seen by capacitors or inductors. By simulating such circuits without regularisation, the simulation fails because they can adopt two or more states at the same time. One solution is to regularise the system's dynamics by introducing suitably located ϵ-parasitic L's and C's considering Tikhonov's Theorem [1]. The regularised circuit now describes a singularly perturbed system and the jump behaviour can be viewed as the limit $\epsilon \to 0$ of the solutions (see [2]). By choosing wrongly located L's and C's, the circuit can be regularised indeed but the determined transient solutions are not reliable. Another problem are the widely spaced time-constants which appear due to the fact that the dynamics of a regularised circuit can be divided in a slow and a fast part, leading to the so-called "time-constant problem" of circuit simulation (see [3]). In this work we propose a geometric approach to overcome these problems and explicitly compute the state space and jump points of an electronic multivibrator.

II. Geometrical Interpretation of Jump Effects in Electronic Circuits

For studying concrete circuits and their generic behaviour as well as conditions for exceptional situations, the descriptive equations from basic physical laws are sufficient. But for describing an entire class of circuits without knowing the details about its circuit structure, a geometric approach is more suitable. There are several monographs including different aspects of the geometric theory of nonlinear dynamical circuits (see e. g. [4], [5], [6], [7] and [8]) but a complete theory and not to mention a concrete application of this theory is still rare. It is known that the state space $\mathcal{S}$ of an electronic circuit can be interpreted as a differentiable manifold [5]. The branch voltages and currents must satisfy the algebraic nonlinear resistive and Kirchhoff constraints. Thus the circuit's state space is defined as the intersection of the Ohmian space $\mathcal{O}$ and the Kirchhoffian space $\mathcal{K}$ by $\mathcal{S} := \mathcal{K} \cap \mathcal{O}$ [9]. The dynamics of a nonlinear dynamic circuit are defined by the set of all solutions of the descriptive differential equations on a sufficient smooth state space $\mathcal{S}$. Therefore the following conditions have to be fulfilled: 1) $\mathcal{S}$ is a smooth manifold. 2) The dynamics can be created on $\mathcal{S}$. The first is a typical or so-called generic condition (for a detailed discussion see [9]), so that in the following we assume $\mathcal{S}$ to be a smooth manifold. The second condition requires the construction of a vector field X on the smooth manifold $\mathcal{S}$. We know, that based on fundamental physical laws, the relationships between currents and voltages of λ capacitors and γ inductors are given by means of differential relations. Therefore these differential equations are formulated in i_L and u_C coordinate planes. Now, we need a projection map π^* to "lift" or "pull-back" the characteristic 2-form G on the state space $\mathcal{S}$ (where $\pi\colon \mathcal{S} \to \mathbb{R}_i^\lambda \oplus \mathbb{R}_u^\gamma$). The characteristic 2-form defines the dynamics of the system. Consequently the vector field X does not exists, if π^* or G is degenerated. E.g., π^* is degenerated if $\mathcal{S}$ contains a fold. A detailed discussion of what types of degeneracy exist can be found in [9]. Now we will discuss how degeneracy and jump effects of a concrete electronic circuit can be analysed. We exclude forced degeneracies, i.e. meshes of capacitors and cut-sets of inductors as well as $L(i) = 0$ or $C(u) = 0$ from our discussion. For this purpose a suitable chart has to be chosen in order to represent $\mathcal{S}$ and the dynamics of a circuit by means of a

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DAE system [10], [11], [12]:

$$\boldsymbol{B}(\boldsymbol{x})\dot{\boldsymbol{x}} = \boldsymbol{g}(\boldsymbol{x},\boldsymbol{y}) \qquad \boldsymbol{g}:\mathbb{R}^k \to \mathbb{R}^n \quad (1)$$

$$\boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) \qquad \boldsymbol{f}:\mathbb{R}^k \to \mathbb{R}^m \quad (2)$$

Here $\boldsymbol{x}$ is the vector corresponding to the capacitor voltages and inductor currents and $\boldsymbol{y}$ is a vector of additional voltages and currents. $\boldsymbol{B}(\boldsymbol{x})$ is a matrix related to the dynamical elements and $\boldsymbol{g}$ represents a nonlinear vector field with respect to $\boldsymbol{x}$ and $\boldsymbol{y}$. The dimension of the embedding space is k, n is the dimension of $\mathcal{S}$, and $m = k - n$ is the codimension. The solution set of the algebraic equations (2) represents the state space $\mathcal{S}$ of the circuit in the chosen chart, whereas the differential equations (1) represent its dynamical behaviour. To obtain the transient response of the circuit, the circuit dynamics have to be integrated by "lifting" the specified dynamics to a vector field on the constraint manifold. As mentioned, lifting may not be possible at points where the projection map has singularities, i.e. points where the local solvability to $\boldsymbol{y}$ is not guaranteed. These points are specified by the following condition:

$$\det\left(\partial_{\boldsymbol{y}}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})\right) = 0 \quad (3)$$

Therefore we assume eq. (3) to be the necessary jump condition (see also [13], [14], [7]). The sufficient jump condition is only given in a heuristic sense: Jump points are those points specified by eq. (3) which are inner points of a set including Lyapunov stable and unstable points. The set of all points fulfilling these two conditions is called "jump-set".

III. Computational Geometric Methods

A. Finding starting points

Since we want to solve the initial value problem of integrating the differential equations restricted to the manifold, we first need a starting point. This is not trivial, since the manifold is given implicitly and no point on the manifold is known a priori. For this purpose, we consider the zero set of the homotopy $H(\boldsymbol{w},\lambda,\boldsymbol{w_0}) = \boldsymbol{f}(\boldsymbol{w}) + (\lambda-1)\boldsymbol{f}(\boldsymbol{w_0})$, [15], [16]. The point $\boldsymbol{w_0}$ lies somewhere in the embedding space and by differentiating we obtain

$$\frac{d\boldsymbol{w}}{d\lambda} = -\boldsymbol{J}^{+}(\boldsymbol{w_0}) \ ,$$

where $\boldsymbol{J}$ is the Jacobian of $\boldsymbol{f}$ and $\boldsymbol{J}^{+}$ is the pseudo-inverse of $\boldsymbol{J}$. Starting from $\boldsymbol{w}(0) = \boldsymbol{w_0}$ we integrate this equation until we arrive at $\boldsymbol{p} := \boldsymbol{w}(1)$ as shown in Fig. 1 curve(1).

B. Parametrisation

Given the starting point $p \in \mathcal{S}$, we can parametrise $\mathcal{S}$ with geodesic coordinates by computing geodesic curves from p as shown in [17]. This parametrisation is univalent until the geodesics reach the cut locus of p [18] and it will allow us to search for the jump-set (cf. Fig. 1: geodesics (2); jump-set (3)). With this parametrisation, each point on the manifold can be described by the length and the starting direction of the respective geodesic curve. The starting direction can be described by $n-1$ angles. Geodesics on submanifolds of $\mathbb{R}^k$ can be described by the condition that their curvature vector projected on the tangential space of the submanifold at the respective point is 0. We can derive a formula for this, where $f_1,\ldots,f_m$ are the components of $\boldsymbol{f}$, and $\boldsymbol{H}_{f_i}$ is the Hessian of f_i. Obviously, for any curve $\boldsymbol{\alpha}(t) = (\boldsymbol{x}(t),\boldsymbol{y}(t))$ on S we have $\boldsymbol{f}(\boldsymbol{\alpha}(t)) = \boldsymbol{0}$. Differentiating two times leads to:

$$\begin{pmatrix} \dot{\boldsymbol{\alpha}}^T\boldsymbol{H}_{f_1}\dot{\boldsymbol{\alpha}} \\ \vdots \\ \dot{\boldsymbol{\alpha}}^T\boldsymbol{H}_{f_m}\dot{\boldsymbol{\alpha}} \end{pmatrix} + \boldsymbol{J}\ddot{\boldsymbol{\alpha}} = 0 \quad (4)$$

Since the space of non-tangential directions is spanned by the gradients of the hypersurfaces f_i we have for some $\boldsymbol{\beta} \in \mathbb{R}^m$:

$$\ddot{\boldsymbol{\alpha}} = \nabla f_1\beta_1 + \cdots + \nabla f_m\beta_m = \boldsymbol{J}^T\boldsymbol{\beta} \Leftrightarrow \boldsymbol{J}\ddot{\boldsymbol{\alpha}} = \boldsymbol{J}\boldsymbol{J}^T\ \boldsymbol{\beta} \quad (5)$$

Solving (5) for $\boldsymbol{\beta}$ and inserting into (4) we can compute geodesics by integrating the differential equation

$$\ddot{\boldsymbol{\alpha}} = -\underbrace{\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1}}_{\boldsymbol{J}^{+}} \begin{pmatrix} \dot{\boldsymbol{\alpha}}^T\boldsymbol{H}_{f_1}\dot{\boldsymbol{\alpha}} \\ \vdots \\ \dot{\boldsymbol{\alpha}}^T\boldsymbol{H}_{f_m}\dot{\boldsymbol{\alpha}} \ . \end{pmatrix}$$

C. Tracing the jump-set

Although the equations given above are valid for any n, we will restrict ourselves to the case $n = 2$. In this case, jump- and hit-set are one-dimensional structures which are more accessible. For higher-dimensional manifolds, we refer to the outlook. We start by tracing geodesics from p in multiple directions. While tracing the geodesics we check the criterion (3) until we find a point on the jump-set. The sufficient Lyapunov condition is in this case identical to a zero crossing of the determinant. From there, we can trace the jump-set $\gamma(s)$ by following its

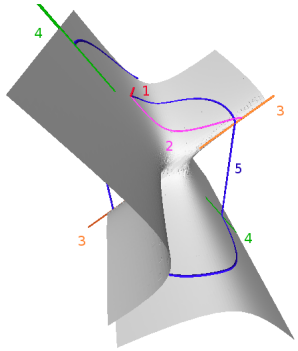

Fig. 1. Manifold defined in 4D-Space by $z^3 - yz^2 - \frac{1}{2}w + \frac{\sqrt{3}}{2}x = 0$ and $\frac{\sqrt{3}}{2}w + \frac{1}{2}x = 0$. The dynamics is rotation-symmetric: $\dot{w} = 0$; $\dot{x} = -y$; $\dot{y} = x$; $\dot{z} = 0$. The jump direction space is spanned by the w and z axes.

tangent vector $\gamma'(s)$. To this end, we differentiate (3) and obtain a linear equation for γ'. Together with $J\gamma' = 0$ and $\|\gamma'\| = 1$, we get a linear equation system that can be solved for γ'.

D. Tracing the dynamics

From a given starting point $p \in S$, the dynamics can be traced by using the following conditions:

i) Every point of the searched curve $\eta(t) = (x(t), y(t))$ has to lie on the manifold, i.e. $f(\eta(t)) = 0$.

ii) The tangential vector $\dot{\eta}(t)$ satisfies $\dot{\eta}(t) = P((g(\eta(t)), 0)^T)$ where P projects the dynamics vector $d = (g(\eta(t)), 0)^T$ into the tangential space $T_{\eta(t)}S$. (Here we assume $B(x)$ to be the identity matrix, so that $\dot{x}(t) = g(x, y)$). We can calculate the projection P using J via $P(d) = d - A(JA)^{-1}Jd$ where the columns of A spans the jump-space.

Our initial intent was to trace the dynamics up until a point, where the "jump condition" (3) holds and then follow the fast part of the dynamics in a jump to another part of the manifold. However, the jump-set cannot be reached by tracing the dynamics, since they are not defined on the jump-set [2]. Because of its singularity, the dynamics increases very fast in the near of the jump-set. Therefore a stopping criterion has to be formulated and implemented.

E. Finding the hit-set

The corresponding "hit-set" is the intersection of the "bundle" of all jump spaces at points of the jump-set and the state space $\mathcal{S}$. Under the natural physical constraints, the energy of capacitors and the charge of inductors is preserved such that the voltage across a capacitance or the current through an inductance have inertia through the jump process and do not change (i.e. the x coordinates do not change during the jump). Another restriction is a fixed state, e.g. by an input voltage which additionally reduces the jump space. Therefore the jump takes place in a subspace parallel to the space given by $\tilde{y}$, which includes all not fixed coordinates of y. Thus the jump direction is predefined and the trajectories "hit" the manifold $\mathcal{S}$. Obviously, for this construction an embedding space is needed. To find a single point $\xi(s)$ of the hit-set corresponding to the actual point $\gamma(s)$ of the jump-set we use a bisection method. This gives us a set of possible points from which we have to choose the one closest to $\gamma(s)$. Another approach is to use a homotopic method. The hit-set can be traced while tracing the jump-set. For this, every tangent vector of the jump-set is translated to the actual position of the hit-set and projected into the tangential space there by using $\xi' = \gamma' - J^+J\gamma'$, see [19]. In Fig. 1 the combination of tracing the dynamic and jumps results in the blue curve (5), while the hit set is shown as the green curve (4).

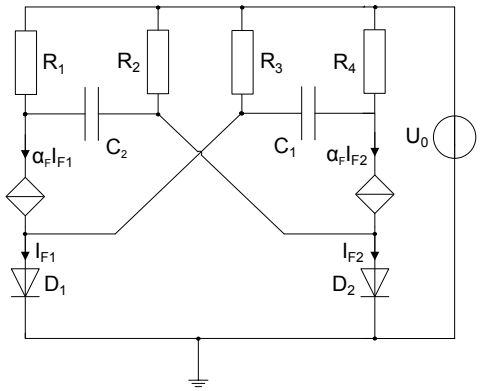

Fig. 2. Multivibrator circuit

IV. Application on an electronic multivibrator

In this section we show our results by applying the derived geometric methods to an electronic multivibrator. To analyse the multivibrator circuit illustrated in Fig. 2, we use the Ebers-Moll model in forward mode. The design parameters are $R_1 = R_4 = 10\Omega$, $R_2 = R_3 = 47k\Omega$, $C_1 = C_2 = 10\mu F$, $I_S = 6.73fA$, $V_T = 26mV$, $\alpha_F = 0.99$ and $U_0 = 9V$. We derive the circuit's algebraic constraints by Kirchhoff's law and get

$$\begin{aligned} 0 &= (U_0 - u_{D1})/R_3 + (U_0 - u_{D1} - u_{C1})/R_4 - \\ &\quad a_F I_S \left(e^{\frac{u_{D2}}{V_T}} - 1\right) - (1 - a_F) I_S \left(e^{\frac{u_{D1}}{V_T}} - 1\right) \\ 0 &= (U_0 - u_{D2})/R_2 + (U_0 - u_{D2} - u_{C2})/R_1 - \\ &\quad a_F I_S \left(e^{\frac{u_{D1}}{V_T}} - 1\right) - (1 - a_F) I_S \left(e^{\frac{u_{D2}}{V_T}} - 1\right), \end{aligned}$$

while the dynamical equations are

$$\begin{aligned} C_1 R_4 \dot{u}_{C1} &= U_0 - u_{C1} - u_{D1} - R_4 a_F I_S \left(e^{\frac{u_{D2}}{V_T}} - 1\right) \\ C_2 R_1 \dot{u}_{C2} &= U_0 - u_{C2} - u_{D2} - R_1 a_F I_S \left(e^{\frac{u_{D1}}{V_T}} - 1\right) . \end{aligned}$$

Here u_{D1} and u_{D2} represent the voltages across the diodes D_1 and D_2 respectively, and u_{C1} and u_{C2} represent the voltages across the capacitances. In Fig. 3 the projected state space in the coordinate system $u_{C1} - u_{C2} - u_{D1}$ is shown. The associated jump-set is represented by the circles and the resulting limit cycle is calculated by solving the differential equations of the regularised multivibrator circuit. As mentioned, the voltages across capacitors cannot jump because of the physical constraints. Consequently the jump takes place in the coordinate plane $u_{D1} - u_{D2}$. Applying the methods of section III yields the state space in the $4D$-coordinate system $(u_{C1} - u_{C2} - u_{D1} - u_{D2})$ (see Fig. 4). Path 1 represents the homotopy path leading to the searched manifold. With a geodesic shooting method (see path 2) the starting point of the jump-set (path 3) is found.

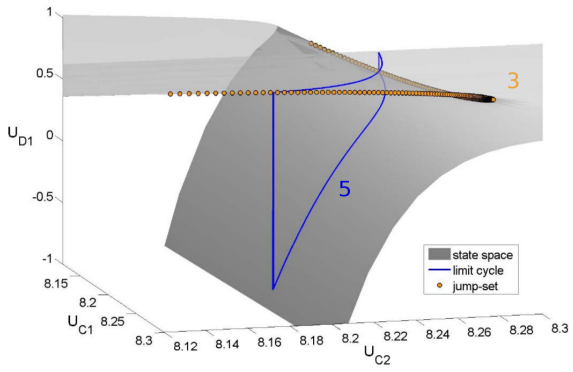

Fig. 3. Projection of the state space in the coordinate system $u_{C1} - u_{C2} - u_{D1}$; limit cycle (5) and calculated jump-set (3)

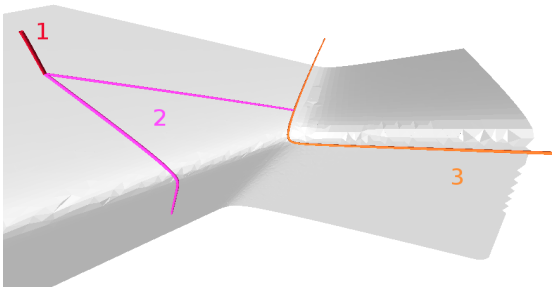

Fig. 4. State space in $4D$-coordinate system ($u_{C1} - u_{C2} - u_{D1} - u_{D2}$) and jump-set (3) of the multivibrator. (1) Homotopy path; (2) Geodesics

Tracing the whole limit cycle without regularisation is still lacking.

V. Conclusion

In this article we have described why certain circuits with jumps are problematic for today's simulation systems and have outlined an alternative to regularisation. We have given a concrete local criterion to check if a circuit's state space manifold exhibits a fold. Together with our differential geometric approach of directly computing significant sets on the state space manifold, i.e. the jump- and hit-set, we can capture the behaviour of the circuit. Using the example of an electronic multivibrator, we have shown that our method works in (until now small) real world applications.

A. Outlook

At the moment, our program is limited to two-dimensional state space manifolds embedded in a space of dimension three or four, resulting in one-dimensional jump- and hit-sets, but an implementation for larger dimensions is currently in development. Moreover, we lack a stopping criterion for tracing the dynamics. One possible approach would be monitoring the speed of the dynamics during the tracing and stopping if it starts to increase faster. We would then choose the nearest point on the jump-set from there and jump to the corresponding point on the hit-set. With our approach it will not be necessary to add a regularisation to get the transient solutions of the circuit.

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